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Hamiltonian structures for the heavy top & plasmas

Our purpose is to discuss the use of symmetry groups in a study of the chaotic dynamics of a heavy top and the Hamiltonian structure of the equations of plasma physics.

THE FREE RIGID BODY

We begin by presenting the equations of a free rigid body. The body is assumed to rotate freely about its center of mass and have a fixed angular velocity vector $\vec{\omega}$ seen by an observer fixed on the body. The body angular momentum $\vec{m}$ is defined by

$$\vec{m} = I\vec{\omega}$$

where $I = \mathrm{diag}(I_1, I_2, I_3)$ is a diagonal matrix coming from the moment of inertia tensor. Assuming that $I_1 > I_2 > I_3$, Euler's equations written in terms of $\vec{m}$ are:

$$\begin{aligned}
\dot{m}_1 &= \frac{I_2 - I_3}{I_2 I_3} m_2 m_3 = a_1 m_2 m_3 \\
\dot{m}_2 &= \frac{I_3 - I_1}{I_1 I_3} m_1 m_3 = a_2 m_1 m_3 \\
\dot{m}_3 &= \frac{I_1 - I_2}{I_1 I_2} m_1 m_2 = a_3 m_1 m_2
\end{aligned} \tag{1}$$

where $a_1 = (I_1 - I_3)/I_2 I_3$ etc. Two basic constants of motion are

Total Angular Momentum ℓ defined by $\ell^2 = m_1^2 + m_2^2 + m_3^2$ (2)

Energy $H(m) = \frac{1}{2} \sum_j m_j^2/I_j$ (3)

Invariance of ℓ^2 in time follows directly from the identity $a_1 + a_2 + a_3 = 0$ and invariance of H follows from the identity $\frac{a_1}{I_1} + \frac{a_2}{I_2} + \frac{a_3}{I_3} = 0$.

The trajectories of (1) are given by intersecting the ellipsoids $H =$ constant from (3) with spheres $\ell =$ constant from (2). For distinct moments of intertia the flow on the sphere has saddle points at $(0,\pm\ell,0)$ and centers at $(\pm\ell,0,0)$, $(0,0,\pm\ell)$. The saddles are connected by four heteroclinic orbits, as indicated in Figure 1.

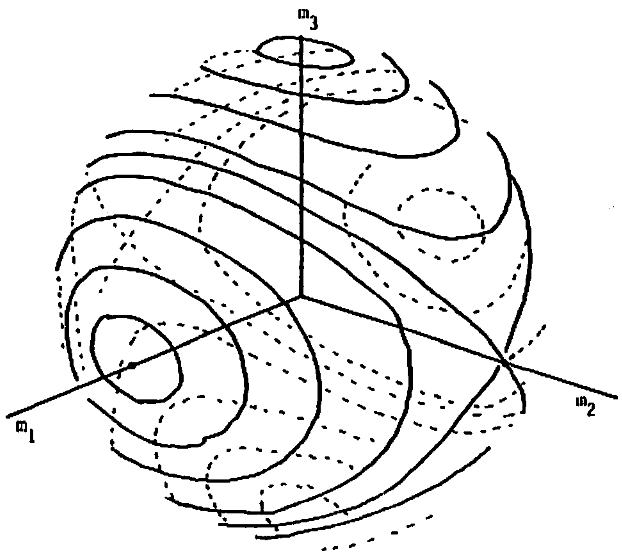

Figure 1. The spherical phase space of the rigid body for fixed total angular momentum $\ell = \sqrt{m_1^2 + m_2^2 + m_3^2}$; $I_1 > I_2 > I_3$

The orbits are explicitly known in terms of elliptic and hyperbolic functions. For example, the heteroclinic orbits lie in the invariant planes.

$$m_3 = \pm\sqrt{\frac{a_3}{a_1}}\; m_1$$

and are given by

$$m_1^+(t) = \pm\ell\sqrt{\frac{a_1}{-a_2}}\ \mathrm{sech}\,(-\sqrt{a_1a_3}\ \ell t)$$

$$m_2^+(t) = \pm\ell\ \tanh\,(-\sqrt{a_1a_3}\ \ell t)$$

$$m_3^+(t) = \pm\ell\sqrt{\frac{a_3}{-a_1}}\ \mathrm{sech}\,(-\sqrt{a_1a_3}\ \ell t)$$

for $m_3 = \sqrt{\frac{a_3}{a_1}}\, m_1$ and by

$$m_1^-(t) = m_1^+(-t)$$

$$m_2^-(t) = m_2^+(-t)$$

$$m_3^-(t) = -m_3^+(-t)$$

for $m = -\sqrt{\frac{a_3}{a_1}}\, m_1$

(Note that $a_1 > 0$, $a_3 > 0$ and $a_2 < 0$). This may be checked by direct computation or by consulting one of the classical texts.

We now introduce a Poisson Bracket for equation (1). Given functions $F,G:R^3 \to R$ we define

$$\{F,G\}(m) = -m \cdot (\nabla F \times \nabla G) \qquad (4)$$

This makes R^3 into a Poisson manifold. This means that on $C^\infty(\mathbb{R}^3)$, the smooth functions on $\mathbb{R}^3$, the bracket $\{\ ,\ \}$ is a Lie algebra ($\{\ ,\ \}$ is real bilinear, skew symmetric and verifies Jacobi's identity) and is a derivation in each of F and G. Later we shall explain where this bracket comes from in group theoretic terms: Here the group is $SO(3)$ and m lives in $SO(3)^* \cong \mathbb{R}^3$, the dual of the Lie algebra of $SO(3)$. By a straightforward calculation one can verify that (1) is equivalent to

$$\dot{F} = \{F, G\} \tag{5}$$

THE HEAVY TOP

We now turn our attention to the heavy top, i.e., a rigid body moving above a fixed point and under the influence of gravity. We let A be a given rotation in $SO(3)$ with corresponding Euler angles denoted (ϕ,ψ,θ). The conjugate momenta are denoted p_ϕ, p_ψ, p_θ so that $(\phi,\psi,\theta,p_\phi,p_\psi,p_\theta)$ coordinatize $T^* SO(3)$. We let m denote the body angular momentum and let $v = A^{-1} k$ where k is the unit vector along the spatial z-axis. We assume that the center of mass is at $(0,0,\ell)$ when A is identity. Coordinates for the vectors (m,v) are most conveniently expressed in the body coordinate system; see Figure 2.

The phase space for the heavy top is $T^*SO(3)$. The system has, however an S^1 symmetry corresponding to rotations about the z-axis. A classical process called reduction enables one to eliminate two of the six variables. (Reduction is described in Arnold [1978] and in Abraham and Marsden [1978].) One gets a reduced space for each value of the angular momentum about the z-axis. One can show (see Marsden, Ratiu and Weinstein [1982] and references therein) that the reduced spaces for the heavy top are sympletically diffeomorphic to T^*S^2 and to a coadjoint orbits for the semi-direct product $SO(3) \times \mathbb{R}^3$; i.e., for the Euclidean group E_3. The Lie algebra of E_3 is denoted $e_3 = SO(3) \times \mathbb{R}^3$. The mapping giving this

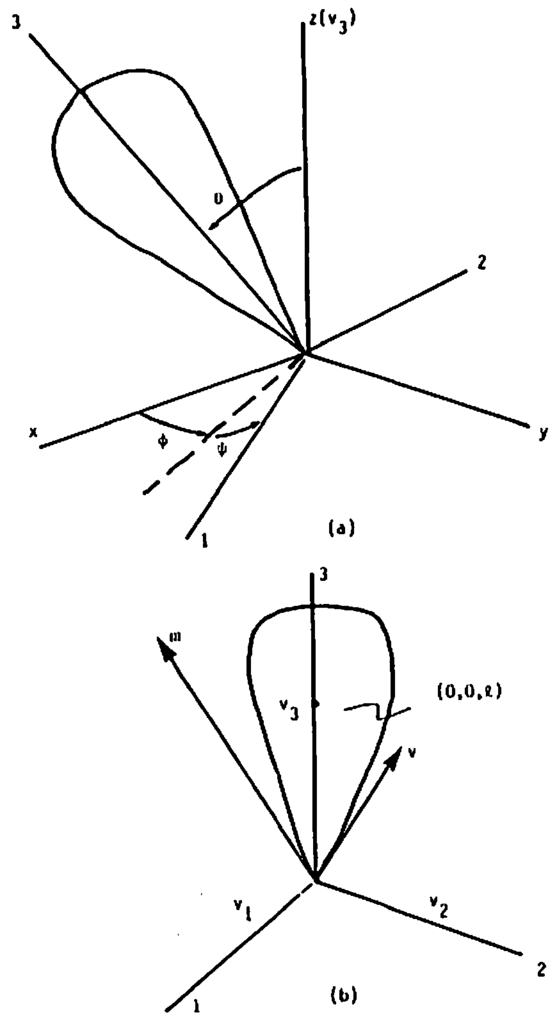

Figure 2. The heavy rigid body, illustration space (x,y,z) and body $(1,2,3)$ coordinates, and the Euler angles (ϕ,ψ,θ).

diffeomorphism is

$$A:(\phi,\psi,\theta,p_\phi,p_\psi,p_\theta) \longmapsto (m,v)$$

Tables in Holmes and Mardsen [1983] give explicit formulae relating these quantities and summarize the relationships between the "Euler angle" spaces and the co-adjoint spaces. A suitable bracket for functions of (m,v) is given by

$$\{F,G\}(m,v) = -m\cdot(\nabla_m F \times \nabla_m G) - v\cdot(\nabla_m F \times \nabla_v G + \nabla_v F \times \nabla_m G) \qquad (6)$$

Like (4), (6) is a special case of a general construction for Lie groups that will be explained below. The Poisson bracket equations are

$$\dot{F} = \{F,G\}$$

where the Hamiltonian H is given by

$$H(m,v) = \tfrac{1}{2}\sum_{j=1}^{3}\frac{m_j^2}{I_j} + Mg\ell v_3$$

and M is the total mass. These equations yield

$$\begin{aligned}
\dot{m}_1 &= a_1 m_2 m_3 - Mg\ell v_2\\
\dot{m}_2 &= a_2 m_1 m_3 + Mg\ell v_1\\
\dot{m}_3 &= a_3 m_1 m_2\\
\dot{v}_1 &= \frac{m_3 v_2}{I_3} - \frac{m_2 v_3}{I_2}\\
\dot{v}_2 &= \frac{m_1 v_3}{I_1} - \frac{m_3 v_1}{I_3}\\
\dot{v}_3 &= \frac{m_2 v_1}{I_2} - \frac{m_1 v_2}{I_1}
\end{aligned} \qquad (7)$$

One can check that these are equivalent to the classical equations of a heavy top; see Holmes and Marsden [1982]. This description of the heavy top in also found in Guillemin and Sternberg [1980].

The foregoing system has $\|v\|$ and $p_\phi = m\cdot v$ as constants of motion. This reflects the conservation law p_ϕ = constant and the preservation of the co-adjoint orbits by the equations. The conditions $\|v\| = 1$ and $m\cdot v = p_\phi$ = constant also provide the identification of the co-adjoint orbit with T^*S^2. Indeed, $\|v\| = 1$ describes the unit sphere S^2 and $m\cdot v = p_\phi$ specifies m as a linear functional on the unit sphere normal to S^2 leaving M restricted to T_vS^2 free. Thus m determines by restriction an element of $T_v^*S^2$. The coordinates θ,ψ are essentially spherical coordinates on S^2.

We discuss the Lagrange top in this framework. For $I_1 = I_2$ we get an additional S^1 symmetry, namely invariance under rotations about the 3-axis. This S^1 action corresponds to the S^1 action of rotation through ψ in the Euler angle picture. Also the momentum map can be readily checked to be just m_3. As with the free rigid body, the Lagrange top has a homoclinic orbit that has an explicit expression in terms of hyperbolic functions.

For the case of a nearly symmetric top, we have:

<u>Theorem (Homes and Mardsen [1983])</u>. <u>If I_1/I_3 is sufficiently large, $I_2 = I_1 + \varepsilon$ and $\varepsilon > 0$ is sufficiently small, then the Hamiltonian system for heavy top has transverse homoclinic orbits (close to the homoclinic orbit for $\varepsilon = 0$) in the Poincaré map for the ψ variable on each energy surface for H = constant in a certain open interval</u>.

The proof of this theorem involves integrating the Poisson bracket $\{H_0,H_1\}$ where $H = H_0 + \varepsilon H_1 + O(\varepsilon^2)$ around the homoclinic orbit for the Lagrange top. These techniques are based on Melnikov [1963].

One concludes that the heavy top close to the symmetric top has no analytic integrals other than the energy and angular momentum about the vertical axis.

Transverse homoclinic orbits implies the presence of Smale horseshoes. Thus, in the motion of a nearly symmetric heavy top, the dynamics is complex, having periodic orbits of arbitrarily high periods and aperiodic orbits embedded in an invariant Cantor set and so the system admits no additional analytical integrals.

Further examples of Hamiltonian systems are provided by Euler's equations for perfect fluids (see Marsden and Weinstein [1982b] and Marsden, Ratiu and Weinstein [1982] for details). In the incompressible case the analogue of m for the free rigid body is the vorticity ω and the bracket is

$$\{F,G\} = \int_\Omega \omega \cdot \left[\frac{\partial F}{\partial \omega}, \frac{\partial G}{\partial \omega}\right] \tag{8}$$

where $\delta F/\delta\omega$ is the functional derivative and $[\ ,\ ]$ is the Lie bracket of vector fields. In the compressible case the bracket on functions of the momentum density $m = \rho u$ and the density ρ is

$$\{F,G\} = \int_\Omega m \left[\frac{\partial F}{\partial m}, \frac{\partial G}{\partial m}\right] + \int \rho \left[\frac{\partial F}{\partial m} \cdot \frac{\nabla \delta G}{\delta \rho} - \frac{\delta G}{\delta m} \cdot \frac{\nabla \delta F}{\delta \rho}\right] . \tag{9}$$

The dynamics of the incompressible case is analogous to that in the rigid body and that of the compressible case parallels that of the heavy top. Other examples are provided by the MHD equations and the plasma equations described below.

LIE-POISSON BRACKETS

We now discuss generalities on bracket structures associated to Lie groups. Let G be a Lie group and $\mathfrak{g}$ be its Lie algebra. For $\xi, \eta \in \mathfrak{g}$, $[\xi,\eta]$ denotes the Lie bracket of ξ and η. Let $\mathfrak{g}^*$ denote the dual space of

of $\mathfrak{g}$. For $F: \mathfrak{g}^* \to \mathbb{R}$ and the variable in $\mathfrak{g}^*$ denoted by μ, define $\delta F/\delta\mu: \mathfrak{g}^* \to \mathfrak{g}$ by

$$DF(\mu)\cdot\nu = \left\langle \nu, \frac{\partial F}{\partial \mu}\right\rangle$$

where $\langle\ ,\ \rangle$ denotes the pairing between $\mathfrak{g}^*$ and $\mathfrak{g}$ and $DF(\mu): \mathfrak{g}^* \to \mathbb{R}$ is the usual Frechet derivative. It is understood that $\partial F/\partial\mu$ is evaluated at the point μ. The Lie-Poisson bracket of two functions $F,G: \mathfrak{g}^* \to \mathbb{R}$ is defined by

$$\{F,G\} = -\left\langle \mu, \left[\frac{\delta F}{\delta \mu}, \frac{\delta G}{\delta \mu}\right]\right\rangle \tag{10}$$

The bracket defines a Poisson structure. This can proved directly or by understanding the relationship of (10) with canonical brackets described below -- see formula (12). From the latter, it is obvious that one obtains a Poisson structure. The bracket (10) is due to S. Lie [1890], p. 235, 294.

The Kirilov-Kostant-Souriau theorem asserts that orbits of the co-adjoint representation in $\mathfrak{g}^*$ are symplectic manifolds. See Arnold [1978] or Abraham and Marsden [1978] for the proof. Thus, $\mathfrak{g}^*$ is a disjoint union of symplectic manifolds. For $F,G: \mathfrak{g}^* \to \mathbb{R}$, a Poisson bracket is thus defined by

$$\{F,G\}(\mu) = \{F|\mathcal{O}_\mu, G|_\mu\}(\mu) \tag{11}$$

where $\mu \in \mathfrak{g}^*$, $\mathcal{O}_\mu$ is the orbit through μ, $F|\mathcal{O}_\mu$ is the restriction of F to $\mathcal{O}_\mu$, and $\{\ ,\ \}$ on the right hand side of (11) is the bracket on $\mathcal{O}_\mu$. This method shows that the bracket $\{F,G\}$ is degenerate; however, it determines a symplectic foliation on each leaf of which it is nondegenerate. The leaves are just the co-adjoint orbits.

Another method of ascertaining the Poisson structure is by extension. Given, $F,G: \mathfrak{g}^* \to \mathbb{R}$ extend them to maps $\hat{F},\hat{G}:T^*G \to \mathbb{R}$ by left invariance. Then using the canonical bracket structure on T^*G, form $\{\hat{F},\hat{G}\}$. Finally, regarding $\mathfrak{g}^*$ as $T_e^*G \subset T^*G$ restrict to $\mathfrak{g}^*$:

$$\{F,G\} = \{\hat{F},\hat{G}\}|_{\mathfrak{g}^*} \tag{12}$$

Formulas (11) and (12) both give (10). (If left invariance is replaced by right invariance, "-" in (10) is replaced by "+".)

Both of the foregoing methods are related by reduction; i.e., the reduced symplectic manifolds for the action of G on T^*G by left translation are the co-adjoint orbits (Marsden and Weinstein [1974]).

MAXWELL's EQUATIONS

In order to review and motivate reduction in more detail we shall now consider the Hamiltonian description of Maxwell's equations. As the configuration space for Maxwell's equations, we take the space $\mathfrak{A}$ of vector fields A on $\mathbb{R}^3$. These are vector potentials related to the magnetic field B by $B = \nabla \times A$. (In the more general situations of Yang-Mills fields, one should replace $\mathfrak{A}$ by the set of connections on a principal bundle). The corresponding phase space is the cotangent bundle $T^*\mathfrak{A}$ with its canonical symplectic structure and a suitable function space topology. Elements of $T^*\mathfrak{A}$ may be identified with pairs (A,Y) where Y is a vector field density on $\mathbb{R}^3$. (We shall not distinguish Y and Ydx). The pairing between A's and Y's is given by integration so that the canonical symplectic structure ω on T^* is given by

$$\omega((A_1,Y_1),(A_2,Y_2)) = \int_{\mathfrak{m}} (Y_2 \cdot A_1 - Y_1 \cdot A_2)dx \; , \tag{13}$$

with the associated canonical Poisson bracket

$$\{F,G\} = \int [\frac{\delta F}{\delta A}\frac{\delta G}{\delta Y} - \frac{\delta F}{\delta Y}\frac{\delta G}{\delta A}]dx \; . \tag{14}$$

Choosing the Hamiltonian

$$H(A,Y) = \tfrac{1}{2}\int |Y|^2 dx + \tfrac{1}{2}\int |\mathrm{curl}\ A|^2 \; , \tag{15}$$

Hamilton's equations are easily computed to be

$$\frac{\delta E}{\delta t} = \mathrm{curl\ curl}\ A \quad \text{and} \quad \frac{\delta A}{\delta t} = Y \; . \tag{16}$$

If we write B for curl A and E for $-Y$, the Hamiltonian becomes the usual field energy

$$\tfrac{1}{2}\int |E|^2 dx + \tfrac{1}{2}\int |B|^2\, dx \tag{17}$$

and the equations (16) imply two of Maxwell's vacuum equations

$$\frac{\delta E}{\delta t} = \mathrm{curl}\ B \quad \text{and} \quad \frac{\delta B}{\delta t} = -\mathrm{curl}\ E \; . \tag{18}$$

The two remaining Maxwell equations will appear as a consequence of gauge invariance. The gauge group G consists of real valued functions on $\mathbb{R}^3$; the group operation is addition. An element $\psi \in G$ acts on $\mathfrak{A}$ by the rule

$$A \to A + \nabla\psi \tag{19}$$

The translation (19) of A extends in a standard way to a canonical transformation ("extended point transformation") of $T^*\mathfrak{A}$ given by

$$(A,Y) \to (A + \nabla\psi, Y) \tag{20}$$

We notice that our Hamiltonian (15) is invariant under the transformation (20). This allows us to use the gauge symmetries to reduce the number of degrees of freedom of our system. The action (20) of G on $T^*\mathcal{O}$ has a momentum map $J:T^*\mathcal{O} \to \mathfrak{g}^*$ where the Lie algebra $\mathfrak{g}$ of G is identified with the real valued functions on $\mathbb{R}^3$. This map J is associated to the symmetry group G in a way that generalizes the way conserved quantities are related to symmetries by Noether's theorem in classical mechanics. We may determine J by a standard formula (Abraham and Marsden [1978]): for $\phi \in \mathfrak{g}$,

$$\langle J(A,Y),\phi \rangle = \int (Y\cdot\nabla\phi)dx = -\int (\operatorname{div} Y)\phi dx$$

Thus we may write

$$J(A,Y) = -\operatorname{div} Y \tag{21}$$

If ρ is an element of $\mathfrak{g}^*$ (the densities on $\mathbb{R}^3$), $J^{-1}(\rho) = \{(A,Y) \in T^*\mathfrak{g} \mid \operatorname{div} Y = -\rho\}$. In terms of E, the condition $\operatorname{div} Y = -\rho$ becomes the Maxwell equation $\operatorname{div} E = \rho$ so we may interpret the elements of $\mathfrak{g}^*$ as charge densities. By a general theorem of Marsden and Weinstein [1974], the reduced manifold $J^{-1}(\rho)/G$ has a naturally induced symplectic structure. Computation leads to the following.

Proposition. The space $J^{-1}(\rho)/G$ can be identified with $\mathrm{Max} = \{(E,B) \mid \operatorname{div} E = \rho, \operatorname{div} B = 0\}$ and the Poisson bracket on Max is given in terms of E and B by

$$\{F,G\} = \int \left(\frac{\delta F}{\delta E}\operatorname{curl}\frac{\delta G}{\delta B} - \frac{\delta G}{\delta E}\operatorname{curl}\frac{\delta F}{\delta B}\right) dx \tag{22}$$

Maxwell's equations with an ambient charge density ρ are Hamilton's equations for

$$H(E,B) = \tfrac{1}{2}\int [|E|^2 + |B|^2]\, dx$$

on the space Max.

The bracket (22) was first introduced, using a different argument, by Born and Infeld [1934].

THE MAXWELL-VLASOV EQUATIONS

We consider a plasma consisting of particles with charge e and mass m moving in Euclidean space $\mathbb{R}^3$ with positions x and velocities v. (For simplicity we consider only one species of particle - the case of several species of particles can be treated in an analogous fashion). Let $f(x,v,t)$ be the plasma density at time t, $E(x,t)$ and $B(x,t)$ the electric and magnetic fields. The Maxwell-Vlasov equations are:

$$\begin{aligned}
&(a)\quad \frac{\partial f}{\partial t} + v\cdot\frac{\partial f}{\partial x} + \frac{e}{m}\left(E + \frac{v\times B}{c}\right)\cdot\frac{\partial f}{\partial v} = 0\\
&(b)\quad \frac{1}{c}\frac{\partial B}{\partial t} = -\operatorname{curl} E\\
&(c)\quad \frac{1}{c}\frac{\partial E}{\partial t} = \operatorname{curl} B - j, \text{ where } j = \frac{e}{c}\int v f(x,v,t)dv\\
&(d)\quad \operatorname{div} E = \rho_f, \text{ where } \rho_f = e\int f(x,v,t)dv\\
&(e)\quad \operatorname{div} B = 0
\end{aligned} \tag{23}$$

Letting $c \to \infty$ leads to the Poisson-Vlasov equation:

$$\frac{\partial f}{\partial t} + v\cdot\frac{\partial f}{\partial x} - \frac{e}{m}\frac{\partial\phi_f}{\partial x}\cdot\frac{\partial f}{\partial v} = 0$$

where

$$\nabla^2\phi_f = \rho_f \tag{24}$$

In what follows we shall set $e = m = c = 1$.

The Hamiltonian for the Maxwell-Vlasov system is

$$H(f,E,B) = \int \tfrac{1}{2}|v|^2 f(x,v,t)\,dx\,dv + \int \tfrac{1}{2}[\,|E(x,t)|^2 + |B(x,t)|^2]\,dx \qquad (25)$$

while that for the Poisson-Vlasov equation is

$$H(f) = \int \tfrac{1}{2}|v|^2 f(x,v,t)\,dx\,dv + \tfrac{1}{2}\int \phi_f(x)\rho_f(x)\,dx \qquad (26)$$

The Poisson bracket for the Maxwell-Vlasov equation is as follows:

$$\begin{aligned}\{F,G\}(f,E,B) &= \int f\left\{\frac{\delta F}{\delta f}, \frac{\delta G}{\delta f}\right\} dx\,dv \\ &+ \int\left\{\frac{\delta F}{\delta E}\,\mathrm{curl}\,\frac{\delta G}{\delta B} - \frac{\delta G}{\delta E}\,\mathrm{curl}\,\frac{\delta F}{\delta B}\right\} dx \\ &+ \int\left\{\frac{\delta F}{\delta E}\cdot\frac{\delta f}{\delta v}\frac{\delta G}{\delta f} - \frac{\delta G}{\delta E}\frac{\delta f}{\delta v}\cdot\frac{\delta F}{\delta f}\right\} dx\,dv \\ &+ \int B\cdot\left[\frac{\partial}{\partial v}\frac{\delta F}{\delta f} \times \frac{\partial}{\partial v}\frac{\delta G}{\delta f}\right] dx\,dv\end{aligned} \qquad (27)$$

The bracket (27) is due to Marsden and Weinstein [1982a] and is based on reduction and an earlier attempt "by hand" by Morrison [1980].

We are now ready to discuss the meaning of the first term $\int f\left\{\frac{\delta F}{\delta f}, \frac{\delta G}{\delta f}\right\} dxdv$ in (27). In the absence of a magnetic field and normalizing the mass, we can identify velocity with momentum. Thus we let $\mathbb{R}^6$ denote the usual position - momentum phase space with co-ordinates $(x_1,x_2,x_3,p_1,p_2,p_3)$ and symplectic structure $\Sigma dx_i \wedge dp_i$. Let $\mathcal{G}$ denote the group of canonical transformations of $\mathbb{R}^6$ which have polynomial growth at infinity in the momentum directions. The Lie algebra $\mathfrak{s}$ of $\mathcal{G}$ consists of the Hamiltonian vector fileds on $\mathbb{R}^6$ with polynomial growth in the momentum directions. We shall identify elements of $\mathfrak{s}$ with their generating functions so that consists of C^∞ functions on $\mathbb{R}^6$ and the left Lie algebra structure is given by $[f,g] = \{f,g\}$, the usual Poisson bracket on phase space. (See Abraham and Marsden [1978]).

The dual space $\mathfrak{s}^*$ can be identified with the distribution densities on R^6 which are rapidly decreasing in the momentum directions. The pairing between $h \in \mathfrak{s}$ and $f \in \mathfrak{s}^*$ is obtained by integration

$$\langle h,f\rangle = \int hf\,dxdp$$

As with any Lie algebra, the dual space $\mathfrak{s}^*$ carries a natural Lie-Poisson structure. In (10) we change "-" to "+" since our system is right invariant. Then with $\mu = f$, (10) becomes

$$\{F,G\}(f) = \langle f, [\tfrac{\delta F}{\delta f}, \tfrac{\delta G}{\delta f}]\rangle = \int f\left\{\frac{\delta F}{\delta f}, \frac{\delta F}{\delta f}\right\} dxdp$$

are required.

The second term in (27) is the bracket for Maxwell's equations which has been previously discussed -- see (22). We consequently turn our attention to the last two terms in (27) which represent coupling or interaction. The Hamiltonian structure for the Maxwell-Vlasov system becomes simple if we choose our variables to be densities on (x,p) space rather than (x,v) space and elements (A,Y) of $T^*\mathcal{A}$. To avoid confusion with densities on (x,v) space, we utilize the notation f_{mom} for densities on (x,p) space.

The Poisson structure on $\mathfrak{s}^* \times T^*\mathcal{A}$ is just the sum of those on $\mathfrak{s}^*$ and $T^*\mathcal{A}$: for functions $\bar{F}$ and $\bar{G}$ of f_{mom}, A and Y, set

$$\begin{aligned}\{\bar{F},\bar{G}\}(f_{mom},A,Y) &= \int f_{mom}\left\{\frac{\partial\bar{F}}{\partial f_{mom}}, \frac{\partial\bar{G}}{\partial f_{mom}}\right\} dxdp + \\ &\int\left(\frac{\partial\bar{F}}{\partial A}\frac{\partial G}{\partial Y} - \frac{\partial\bar{G}}{\partial A}\frac{\partial\bar{F}}{\partial Y}\right) dx\end{aligned} \qquad (28)$$

and the Hamiltonian is just (25) written in terms of these variables. Using the classical relation between momentum and velocity, $p = v + A$, we have

$$H(f_{mom},A,Y) = \frac{1}{2}\int |p - A(x)|^2 f_{mom}(x,p)\,dxdp + \frac{1}{2}\int (|Y|^2 + |\text{curl } A|^2)dx$$

We observe that there is no coupling in the symplectic structure but there is coupling between f_{mom} and A in the first term of (29).

Theorem. The evolution equations $\dot{F} = \{F,H\}$ for a function F on $s^* \times T^*$ with H given by (29) and $\{\ \}$ by (28) are the equations (23a,b,c) with (23b) replaced by $\frac{\partial A}{\partial t} = Y$.

The proof of this theorem is a straightforward verification. The constraints can, as in Morrison [1980], be regarded as subsidiary equations which are consistent with the evolution equations. Equations (23b and e) hold since $B = \text{curl } A$. We will now show that equation (23d) expresses the fact that we are on the zero level of the momentum map generated by the gauge transformations. The corresponding reduced space decouples the energy, while coupling the symplectic structure.

The work of Weinstein [1978] on the equations of motion for a particle in a Yang-Mills field uses the following general set-up. Let $\pi: P \to M$ be a principle G-bundle and Q a Hamiltonian G-space (or a Poisson manifold which is a union of hamiltonian G-spaces). Then G acts on T^*P and on Q, so it acts on $Q \times T^*P$ (with the product symplectic structure). This action has a momentum map J and so may be reduced at 0:

$$(Q \times T^*P)_0 = J^{-1}(0)/G \tag{30}$$

The reduced manifold (30) carries a symplectic (or Poisson, if Q was a Poisson manifold) structure naturally induced from those of Q and T^*P.

To obtain the phase space for an elementary particle in a Yang-Mills field one chooses P to be a G-bundle over 3-space M and Q a co-adjoint

orbit for G (the internal variables). The Hamiltonian is constructed using a connection (i.e., a Yang-Mills field) for P. In the special case of electromagnetism, $G = S^1$ and $Q = \{e\}$ is a point.

For the Vlasov-Maxwell system we choose our gauge bundle to be

$$P = \mathcal{O}\!\ell \to M$$

where $M = \{B | \text{div } B = 0\}$, with G the gauge group described in the previous section. As in §3, let $\mathcal{G}$ denote the group of canonical transformations of $T^*M (= \mathbb{R}^6)$. We can let Q be either the symplectic manifold $T^*\mathcal{G}$ or the Poisson manifold s^*. It is a little more direct work with s^*, so we shall do this.

We wish to specify an action of G on s^* which, when combined with the action (20) on $T^*\mathcal{O}\!\ell$, will leave the Hamiltonian (29) invariant. A natural choice is to let $\psi \in G$ act by the (linear) map

$$f_{mom} \to f_{mom} \circ \tau_{\nabla\psi} \tag{31}$$

where $\tau_{-\nabla\psi}: \mathbb{R}^6 \to \mathbb{R}^6$ is the "momentum translation map" defined by

$$\tau_{\nabla\psi}(x,p) = (x, p - \nabla\psi(x)). \tag{32}$$

It is easy to verify that $\tau_{-\nabla\psi}$ is a canonical transformation, so it preserves the ordinary Poisson bracket on $\mathbb{R}^6$. It follows that the map (31) preserves the Poisson structure on s^*. A simple calculation gives:

Lemma. The action of G on s^* defined by (31) and (32) has a momentum map $J: s^* \to \mathfrak{g}^*$ given by

$$\langle J(f_{mom}), \phi\rangle = -\int f_{mom}(x,p)\phi(x)\,dxdp \tag{33}$$

i.e.

$$J(f_{mom}) = -\int f_{mom}(x,p)\, dp \tag{34}$$

The right hand side of (34) is a density on $\mathbb{R}^3$ which we may denote by ρ_{mom}.

Now we define the action of G on the product $s^* \times T^*\mathfrak{A}$ by combining (31) and (20), i.e. $\psi \in G$ maps

$$(f_{mom}, A, Y) \to (f_{mom} \circ \tau_{-\nabla\phi},\ A + \nabla\phi, Y). \tag{35}$$

Combining equations (21) and (34) gives:

Lemma. The momentum map $J: s^* \times T^*\mathfrak{A} \to \mathfrak{g}^*$ for the action (35) is given by:

$$J(f_{mom}, A, Y) = -\int f_{mom}(x,p)\, dp - \operatorname{div} Y. \tag{36}$$

We may now describe the reduced Poisson manifold in terms of densities $f(x,v)$ defined on position-velocity space.

Proposition. The reduced manifold $(s^* \times T^*\mathfrak{A})_0 = J^{-1}(0)/G$ may be identified with the Maxwell-Vlasov phase space

$$MV = \{(f,B,E) \mid \operatorname{div} B = 0 \text{ and } \operatorname{div} E = \int f(x,v)\, dv.\}$$

Proof. To each (f_{mom}, A, Y) in $J^{-1}(0)$ we associate the triple (f,B,E) in MV where

$$f(x,v) = f_{mom}(x, v + A(x)),\ B = \operatorname{curl} A, \text{ and } E = -Y.$$

The condition $J(f_{mom}, A, Y) = 0$ is equivalent, by (36), to the Maxwell equation $\operatorname{div} E = \int f(x,v)\, dv$ in the definition of MV. It is easy to check that elements of $J^{-1}(0)$ are associated to the same (f,B,E) if and only if they are related by a gauge transformation (35), so our association gives a 1-1 correspondence between $J^{-1}(0)/G$ and MV. ∎

By the general theory of reduction, MV inherits a Poisson structure from the one on $s^* \times T^*$. Since the Hamiltonian (29) is invariant under G, it follows that the Maxwell-Vlasov equations are a Hamiltonian system on MV with respect to this structure. We can compute the explicit form of the inherited Poisson structure in the variables (f,B,E). In fact, a direct calculation using the chain rule shows that (28) becomes (27). This is how one arrives at the following result.

Theorem. The bracket (27) makes (MV) into a Poisson manifold. The Maxwell-Vlasov equations are equivalent to the evolution equations $\dot F = \{F,H\}$ on (MV), where H is given by (25).

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REFERENCES.

1. R. Abraham and J. Marsden [1978]. Foundations of Mechanics, Second Edition, Addition-Wesley.
2. V. Arnold [1978]. Mathematical methods of classical mechanics. Graduate Texts in Math. No. 60 Springer Verlag, New York.
3. M. Born and L. Infeld [1935]. On the quantization of the new field theory, Proc. Roy. Soc. A. 150, 141.

4. P.J. Holmes and J.E. Marsden [1983]. Horseshoes and Arnold diffusion for Hamiltonian systems on Lie groups. Ind. Univ. Math. J. (to appear).

5. J. Marsden, T. Ratiu and A. Weinstein [1982]. Semi-direct products and reduction in mechanics (preprint).

6. J. Marsden and A. Weinstein [1974]. Reduction of symplectic manifolds with symmetry, Rep. Math. Phys., 5, pp. 121-130.

7. J. Marsden and A. Weinstein [1982a]. The Hamiltonian structure of the Maxwell-Vlasov equations, Physica D4, 394-406.

8. J. Marsden and A. Weinstein [1982b]. Coadjoint orbits, vortices and Clebsch variables for incompressible fluids, Physica D. (to appear).

9. P.J. Morrison [1980]. The Maxwell-Vlasov equations as a continuous Hamiltonian system, Phys. Lett. 80A, 383-386.

10. V.K. Melnikov [1963]. On the stability of the center for time period perturbations, Trans. Moscow Math. Soc. 12, 1-57.

11. A. Weinstein [1978]. A universal phase space for particles in Yang-Mills fields, Lett. Math. Phys. 2, 417-420.

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Use of steepest descent for systems of conservation equations

Many systems of conservation equations have the form

$$u_t = \nabla\cdot F(u,\nabla u)\ , \tag{1}$$

$u: [0,T] \times \Omega \to R^k$, $\Omega \subset R^n$. Often, however, a more complicated form is found:

$$Q(u)_t = \nabla\cdot F(u,\nabla u) \tag{2}$$

$$S(u) = 0,$$

$u: [0,T] \times \Omega \to R^{k+q}$, $\Omega \subset R^n$, $Q: R^{k+q} \to R^k$, $S: R^{k+q} \to R^q$, $F: R^{k+q} \times R^{n(k+q)} \to R^{nk}$. Examples are found in many places (cf [1], [6]).

Some background for the present work is found in [2], [3].

Often problem (2) can be changed into (1) by using the condition $S(u) = 0$ to eliminate q of the unknowns and by using some change of variables to convert the term $(Q(u))_t$ into the form u_t. However since applications often seem to lead to problems naturally expressed in form (2), it seems worthwhile to study (2) directly. We consider homogeneous boundary conditions on $\delta\Omega$ together with initial conditions.

We introduce a strategy for a time-stepping procedure. We at first discretize in time only. Take w for a time-slice of a solution at a time t_0. We seek an estimate v at time $t_0+\delta$ by seeking to minimize